On "Effective potential for a covariantly constant gauge field in curved spacetime"

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Abstract

We extend recent work by Elizalde et al. to incorporate curvatures which are not small and backgrounds which are not just $S^2 \times R^2, S^1 \times S^1 \times R^2$. Some possible problems in their paper is also pointed out.

Introduction

The calculation of effective potentials for gauge fields to one loop order and beyond is certainly an important task. Recently Elizalde, Odintsov and Romeo, [1], has carried out such a calculation for a covariantly constant $SU(N)$-field on a curved background of the form $S^2 \times R^2, S^1 \times S^1 \times R^2$, in the limit of small curvature (i.e. large radii). Elsewhere we have considered the same problem in a more general setting, [2], namely a general Yang-Mills field on a curved background with not too violently varying curvature. We believe that this method solves some of the problems faced by Elizalde et al. A fuller investigation of this is in the making and will be submitted shortly, [3]. [1]

Elizalde has also given a thorough discussion of the techniques underlying his work with Odintsov and Romeo in [4].

[1] In the first of these two papers, a mistake has crept in: a term of the form $\delta^a_b R^n_m$, where R^n_m is the Ricci tensor was missing, this mistake has been corrected in [3], which also contains many applications.

As shown in [2], the effective Lagrangian for a general Yang-Mills field on a general curved background can be written as

$$\begin{aligned} V_{\text{eff}}(A) &= (4\pi)^{-2}\text{Tr}\,\Bigg(-\frac{g^6}{128}\mathcal{A}^2\ln\,\frac{g^2}{4}\mathcal{A}+\frac{3g^6}{256}\mathcal{A}^2 \\ &\qquad -\frac{1}{2}\left(\ln\,\frac{g^2}{4}\mathcal{A}\right)\,\mathcal{B}-\frac{16}{3g^4}\mathcal{A}^{-1}\mathcal{C}\Bigg)-\frac{g^2}{4}F^a_{mn}F_a^{mn} \quad (1) \end{aligned}$$

where $\mathcal{A},\mathcal{B},\mathcal{C}$ are matrices defined by

$$\begin{aligned} \mathcal{A}^{m(a)}_{n(b)} &= \left(\partial_p\mathcal{E}^{mp}_n+\frac{3}{4}\mathcal{E}^{mp}_k\mathcal{E}^k_{np}\right)\delta^a_b+gf_b{}^a{}_c(\partial_nA^{mc}-\partial^mA^c_n)+ \\ &\qquad \frac{1}{2}\delta^m_n g^2 f_{ebc}f_d{}^a{}_c A^e_p A^{pd}+\delta^a_b R^m_n \quad (2) \\ \mathcal{B}^{m(a)}_{n(b)} &= \Box_0\mathcal{A}^{m(a)}_{n(b)}\equiv\eta^{pq}\partial_p\partial_q\mathcal{A}^{m(a)}_{n(b)} \quad (3) \\ \mathcal{C}^{m(a)}_{n(b)} &= (\partial_p\mathcal{A}^{m(a)}_{k(c)})(\partial^p\mathcal{A}^{k(c)}_{n(b)}) \quad (4) \end{aligned}$$

where

$$\mathcal{E}^{mp}_n=(\partial_n e^{m\mu}-\partial^m e^\mu_n)\,e^p_\mu \quad (5)$$

and $\partial_m = e^\mu_m\partial_\mu$. The quantities e^μ_m are vierbeins, $g^{\mu\nu}=e^\mu_m e^\nu_n\eta^{mn}$, and $A^a_m=e^\mu_m A^a_\mu$ etc.

In the case studied in [1] with

$$A^a_\mu=-\frac{1}{2}F^a_{\mu\nu}x^\nu \quad (6)$$

covariantly constant ($F^a_{\mu\nu}=const$) we get the the gauge part of these quantities to be

$$\begin{aligned} \mathcal{A}^{m(a)}_{n(b)} &= -\frac{1}{2}gf_b{}^a{}_c\left(e^\mu_n(\partial_\mu e^\nu_p)\eta^{mp}F^c_{\nu\rho}x^\rho-e^\nu_p(\partial_\nu e^\mu_n)\eta^{mp}F^c_{\mu\rho}x^\rho+e^\mu_n e^\nu_p\eta^{mp}F^c_{\mu\nu}\right) \\ &\quad +\frac{1}{8}g^2\delta^m_n f_{ebc}f_d{}^{ac}g^{\mu\nu}F^d_{\mu\rho}F^d_{\nu\sigma}x^\rho x^\sigma+\text{curvature terms} \quad (7) \\ \mathcal{B}^{m(a)}_{n(b)} &= -\frac{1}{2}gf_b{}^a{}_c\eta^{rs}\eta^{mp}e^\rho_r\left[\partial_\rho(e^\sigma_s\partial_\sigma(e^\mu_n\partial_\mu e^\nu_p))F^c_{\nu\kappa}x^\kappa-\partial_\rho(e^\sigma_s\partial_\sigma(e^\nu_p\partial_\nu e^\mu_n))F^c_{\mu\kappa}x^\kappa\right] \\ &\quad -\frac{1}{2}gf_b{}^c{}_c\eta^{rs}\eta^{mp}e^\rho_r\left[\partial_\rho(e^\sigma_s e^\mu_n\partial_\mu e^\nu_p)F^c_{\nu\rho}-\partial_\rho(e^\sigma_s e^\nu_p\partial_\nu e^\mu_n)F^c_{\mu\rho}-2\partial_\rho(e^\sigma_s\partial_\sigma(e^\mu_n e^\nu_p))F^c_{\mu\nu}\right] \\ &\quad +\frac{1}{8}g^2\eta^{rs}\delta^m_n f_{ebc}f_d{}^{ac}F^e_{\mu\lambda}F^d_{\nu\kappa}e^\rho_r\partial_\rho(e^\sigma_s\partial_\sigma(g^{\mu\nu}x^\lambda x^\kappa))+\text{curvature terms} \quad (8) \end{aligned}$$

$$
\begin{aligned}
\mathcal{C}^{m(a)}_{n(b)} &= g^{\rho\sigma}\Big[(\partial_\rho(e^\mu_k\partial_\mu e^\nu_p)\eta^{mp}A^c_\nu - \frac{1}{2}e^\mu_k(\partial_\mu e^\nu_k)\eta^{mp}F^c_{\nu\rho} \\
&\quad -\partial_\rho(e^\nu_p\partial_\nu e^\mu_k)\eta^{mp}A^c_\mu + \frac{1}{2}e^\nu_p(\partial_\nu e^\mu_k)\eta^{mp}F^c_{\mu\rho} + \partial_\rho(e^\mu_k e^\nu_p)\eta^{mp}F^c_{\mu\nu}\Big] \times \Big[\partial_\sigma(e^\lambda_n\partial_\lambda e^\kappa_q)\eta^{kq}A^d_\kappa \\
&\quad -\frac{1}{2}e^\lambda_n(\partial_\lambda e^\kappa_q)\eta^{kq}F^d_{\kappa\sigma} - \partial_\sigma(e^\kappa_q\partial_\kappa e^\lambda_n)\eta^{kq}A^d_\lambda + \frac{1}{2}e^\kappa_q(\partial_\kappa e^\lambda_n)\eta^{kq}F^d_{\lambda\sigma} + \partial_\sigma(e^\lambda_n e^\kappa_q)\eta^{kq}F^d_{\lambda\kappa}\Big] \\
&\quad +\frac{1}{64}g^4\delta^m_n f_{ebc}f_{e'b'c'}f_d{}^{b'c}f_{d'}{}^{ac'}F^e_{\mu\lambda}F^{e'}_{\mu'\lambda'}F^d_{\nu\kappa}F^{d'}_{\nu'\kappa'}g^{\rho\sigma}\partial_\rho(g^{\mu'\nu'}x^{\lambda'}x^{\kappa'})\partial_\sigma(g^{\mu\nu}x^\lambda x^\kappa) \\
&\quad +\frac{1}{4}g^2\delta^m_n f_{ebc}f_d{}^{ac}F^e_{\mu\lambda}F^d_{\nu\kappa}g^{\rho\sigma}\partial_\rho(g^{\mu\nu}x^\lambda x^\kappa) \times \partial_\sigma\Big[e^\epsilon_k A^b_\phi(\partial_\epsilon e^\phi_b)\eta^{kp} - e^\phi_p A^b_\epsilon(\partial_\phi e^\epsilon_k)\Big] \\
&\quad +\text{curvature terms} \qquad (9)
\end{aligned}
$$

Now, for the spacetimes considered by Elizalde et al., [1], namely $S^2\times R^2$, $S^1\times S^1\times R^2$, the vierbeins can be chosen to only depend on the radii, and hence $\mathcal{E}^{mp}_n \equiv 0$, the only curvature dependency then comes from the Ricci tensor. Since this is a constant, $R_{\mu\nu} \propto \rho^{-2}$ (where ρ is the radius of S^2, and the result for the torus $S^1\times S^1$ is slightly more complicated but of the same nature), only $\mathcal{A}$ will contain this, while $\mathcal{B}, \mathcal{C}$ will be curvature independent. In these cases, furthermore, $\mathcal{B}, \mathcal{C}$ become diagonal (except, perhaps, in colour space). Explicitly,

$$
\begin{aligned}
\mathcal{A}^{m(a)}_{n(b)} &= -\frac{1}{2}gf_b{}^a{}_c\eta^{mp}e^\mu_n e^\nu_p F^c_{\mu\nu} + \frac{1}{8}g^2\delta^m_n g^{\mu\nu}f_{ebc}f_d{}^{ac}F^e_{\mu\rho}F^d_{\nu\sigma}x^\rho x^\sigma + \delta^a_b R^m_n \qquad (10) \\
\mathcal{B}^{m(a)}_{n(b)} &= \frac{1}{8}g^2\delta^m_n f_{ebc}f_d{}^{ac}F^e_{\mu\lambda}F^d_{\nu\kappa}g^{\rho\sigma}g^{\mu\nu}(\delta^\lambda_\rho\delta^\kappa_\sigma + \delta^\lambda_\sigma\delta^\kappa_\rho) \qquad (11) \\
\mathcal{C}^{m(a)}_{n(b)} &= \frac{1}{64}g^4\delta^m_n g^{\rho\sigma}f_{e'b'c'}f_{d'}{}^{ac'}f_{ebc}f_d{}^{b'c}g^{\mu\nu}g^{\mu'\nu'} \times \\
&\quad F^{e'}_{\mu'\lambda'}F^{d'}_{\nu'\kappa'}F^e_{\mu\lambda}F^d_{\nu\kappa}(\delta^{\lambda'}_\rho x^{\kappa'} + \delta^{\kappa'}_\rho x^{\lambda'})(\delta^\lambda_\sigma x^\kappa + \delta^\kappa_\sigma x^\lambda) \qquad (12)
\end{aligned}
$$

With the explicit choice of the the field strength tensor used by Elizalde et al., (the first two coordinates being S^2, the last two R^2)

$$
F^a_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & H^a \\ 0 & 0 & -H^a & 0 \end{pmatrix}
$$

and using that the Ricci tensor is simply

$$R_{\mu\nu} = \begin{pmatrix} \rho^{-2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{13}$$

we get $\mathcal{A}$ to be block diagonal

$$\mathcal{A} = \begin{pmatrix} X+\rho^{-2} & 0 & 0 & 0 \\ 0 & X+1 & 0 & 0 \\ 0 & 0 & X & h \\ 0 & 0 & -h & X \end{pmatrix} \equiv \begin{pmatrix} \tilde{X} & 0 \\ 0 & a \end{pmatrix} \tag{14}$$

where (x, y are the coordinates of R^2)

$$X = \frac{1}{8} g^2 g^{\mu\nu} f_{ebc} f_d{}^{ac} F^e_{\mu\rho} F^d_{\nu\sigma} x^\rho x^\sigma = \frac{1}{8} g^2 f_{ebc} f_d{}^{ac} H^e H^d (y^2 - x^2) \tag{15}$$

$$h = -\frac{1}{2} g f_b{}^a{}_c H^c \tag{16}$$

The logarithm of $\mathcal{A}$ appears in the result for the effective action, since $\mathcal{A}$ is block diagonal the calculation of this quantity is straightforward, it is

$$\ln \mathcal{A} = \begin{pmatrix} \ln \tilde{X} & 0 \\ 0 & b \end{pmatrix} \tag{17}$$

with $a = e^b$. Now, any 2×2 matrix can be expanded on the Pauli matrices, $a = a_0 1_2 + a_i \sigma^i$, using the algebraic properties of these we then get

$$b = \begin{pmatrix} \ln X - \ln \cos h & ih \\ -ih & \ln X - \ln \cos h \end{pmatrix} \tag{18}$$

As for Elizalde and coworkers, the effective action gets an imaginary part from the logarithmic term.
The explicit form for the remaining matrices turn out to be

$$\mathcal{B}^{m(a)}_{n(b)} = \delta^m_n \frac{1}{2} g^2 f_{ebc} f_d{}^{ac} H^e H^d \equiv \delta^m_n h_2 \tag{19}$$

$$\mathcal{C}^{m(a)}_{n(b)} = \delta^m_n \frac{g^4}{16} f_{ebc} f_{e'b'c'} f_d{}^{b'c} f^{ac'}_{d'} H^e H^{e'} H^d H^{d'} (y^2 - x^2) \equiv \delta^m_n h_4 \tag{20}$$

Inserting all of this into our general formula (1) we then finally arrives at

$$
\begin{aligned}
V_{\text{eff}} &= (4\pi)^{-2}\mathrm{tr}\Bigg\{-\frac{g^6}{128}\Bigg((X^2+\rho^{-2})^2\ln\frac{g^2}{4}(X^2+\rho^{-2}) \\
&\quad +(X+1)^2\ln\frac{g^2}{4}(X+1)+2(X^2-h^2)\ln\frac{g^2}{4}X \\
&\quad -2(X^2-h^2)(\ln\cos\frac{g^2}{4}h+\frac{1}{2}ig^2h^2X)\Bigg)+\frac{3g^6}{256}\left(X^2+\rho^{-2})^2+3X^2-2h^2+2X+1\right) \\
&\quad -\frac{1}{2}h_2\left(\ln\frac{g^2}{4}(X+\rho^{-2})+2\ln\frac{g^2}{4}X-2\ln\cos\frac{g^2}{4}h+\ln\frac{g^2}{4}(X+1)\right) \\
&\quad -\frac{16}{3g^4}\frac{h_4}{(h^2+X^2)(X+1)(X+\rho^{-2})}\Big((h^2+X^2)(2X+1+\rho^{-2})+2X(X+\rho^{-2})(X+1)\Big)\Bigg\} \\
&\quad -\frac{g^2}{2}H^aH_a
\end{aligned}
\tag{21}
$$

where the trace is over gauge algebra indices (the result is valid for an arbitrary Lie algebra).
The result which Elizalde et al. finds is (Ω being the volume of the two-sphere)

$$
\frac{\Gamma}{\Omega}=a_0(gH)^2\left[\frac{11}{48\pi^2}\left(\ln\frac{gH}{\mu'^2}-\frac{1}{2}\right)-i\frac{1}{8\pi}\right]+\frac{1}{4\pi^2}\frac{gH}{\rho^2}\left[-\frac{a_0+a_1}{2}\ln 2+ia_1\frac{\pi}{2}\right]
$$

where a_0, a_1 are coefficients in the Schwinger-DeWitt expansion of the heat kernel of the Laplacean on S^2, ($a_0 = 1, a_1 = -1/3$), and where the gauge group has been chosen to be $SU(2)$. A number of approximations have been made here, first of all the calculation is linear in the curvature and secondly it is only valid for ρ large. This latter comment, however, does not refrain the authors of [1] from studying $1 \leq \rho \leq 10$, a regime in which the approximation $\rho \gg 1$ certainly cannot be said to hold. With this they claim to find a critical point at $\rho_c \sim 2$. One should note that the calculation put forward here does not suffer from these problems.
In our calculation we have used the freedom in renormalisation to fix $\mu = 1$. The form of the result by Elizalde and coworkers is $(H^2)(\ln H - 1/2) + RH$ with $R = \rho^{-2}$, these two terms are, up-to a finite renormalisation, the same as the lowest order terms in our formula. The remaining terms are due to non-linear terms in the curvature and cannot therefore be found by the mode summation of Elizalde, Odintsov and Romeo.

Conclusion

We have shown how the effective action for a covariantly constant Yang-Mills field on a simple background $S^2 \times R^2$ can be found, including terms non-linear in the curvature, thereby generalising the result of Elizalde, Odintsov and Romeo. Furthermore, as our method is not based on an explicit mode summation, it is applicable to spacetimes in which one does not know the explicit form of the eigenvalues of the Laplace operator for spin one (i.e. almost all spacetimes). An important example of such a spacetime manifold, mentioned also by Elizalde et al., is de Sitter S^4. Since our approach only needs the vierbein (and through them the Ricci tensor etc.), this spacetime is actually within reach of the method proposed here. Manifolds with non-constant curvature can also be treated, which means that all manifolds of physical interest should be tractable.

Our result is also capable of handling non-covariantly constant field configurations and to calculate curvature induced mean-fields as shown in [2, 3], where also phase transitions are treated. Further research into this is in progress.

References

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